

# RANDOM GAPS

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## Abstract

It is proved that there exists an  $(\omega_1, \omega_1)$  Souslin gap in the Boolean algebra  $(L^0(\nu)/\text{Fin}, \subseteq_{\text{ae}}^*)$  for every nonseparable measure  $\nu$ . Thus a Souslin, also known as destructible,  $(\omega_1, \omega_1)$  gap in  $\mathcal{P}(\mathbb{N})/\text{Fin}$  can always be constructed from uncountably many random reals. We explain how to obtain the corresponding conclusion from the hypothesis that Lebesgue measure can be extended to all subsets of the real line (RVM).

## 1 Introduction

A *pregap* in a Boolean algebra  $(\mathcal{B}, \leq)$  is an orthogonal pair  $(A, B)$  of subsets of  $\mathcal{B}$ , i.e.

- (i)  $a \wedge b = 0$  for all  $a \in A$  and  $b \in B$ ,

and it is a *gap* if additionally there is no element  $c$  of  $\mathcal{B}$  such that

- (ii)  $a < c$  for all  $a \in A$ , and  $b < -c$  for all  $c \in B$ .

Such an element  $c$  is said to *interpolate* the pregap. A *linear pregap* is a pregap  $(A, B)$  where both  $A$  and  $B$  are linearly ordered by  $\leq$ , and for a pair of linear order types  $(\varphi, \psi)$ , a  $(\varphi, \psi)$  *pregap* in a Boolean algebra  $(\mathcal{B}, \leq)$  is a linear pregap  $(A, B)$  where  $\text{otp}(A, \leq) = \varphi$  and  $\text{otp}(B, \leq) = \psi$ . Thus  $(A, B)$  is a  $(\varphi, \psi)$  *gap* if it is a  $(\varphi, \psi)$  pregap for which no element of  $\mathcal{B}$  can be used to extend  $(A, B)$  to a  $(\varphi + 1, \psi)$  pregap or a  $(\varphi, \psi + 1)$  pregap.

We let  $\mathbb{N}$  denote the set  $\{0, 1, 2, \dots\}$  of nonnegative integers, and  $\mathcal{P}(\mathbb{N})$  the power set of  $\mathbb{N}$  quasi ordered by  $a \subseteq^* b$  if  $a \setminus b$  is finite.  $\mathcal{P}(\mathbb{N})/\text{Fin}$  denotes the equivalence classes of  $\mathcal{P}(\mathbb{N})$  modulo the equivalence relation of finite set difference, with the induced partial ordering  $[a] \subseteq^* [b]$  if  $a \subseteq^* b$ . One can find results on gaps in the Boolean algebra  $(\mathcal{P}(\mathbb{N})/\text{Fin}, \subseteq^*)$  dating back to the second half of the 19th century, including a basic result (du Bois-Reymond) appearing in 1873, and Hadamard's Theorem (1894) that there are no  $(\omega, \omega)$

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gaps in  $\mathcal{P}(\mathbb{N}) / \text{Fin}$ . Indeed, one of the major achievements in early Set Theory was Hausdorff’s construction (1936) of an  $(\omega_1, \omega_1)$  gap in  $\mathcal{P}(\mathbb{N}) / \text{Fin}$  (he actually first constructed in (1909) an  $(\omega_1, \omega_1)$  gap in a different structure, that has a simple translation to a gap in  $\mathcal{P}(\mathbb{N}) / \text{Fin}$ ). See Scheepers (1993) for the history of gaps.

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## 1.2 Destructibility

While being a pregap in  $\mathcal{P}(\mathbb{N}) / \text{Fin}$  is absolute between any ‘reasonable’ models—e.g. transitive models of some large enough fragment of ZF—, the property of being a gap is not. For example, if  $(A, B)$  is an  $(\omega_1, \omega_1)$  gap in  $\mathcal{P}(\mathbb{N}) / \text{Fin}$  and  $\mathcal{Q}$  is a poset which collapses  $\aleph_1$ , then by Hadamard’s theorem, which generalizes to any limit ordinal of countable cofinality, forcing with  $\mathcal{Q}$  must introduce an element of  $\mathcal{P}(\mathbb{N}) / \text{Fin}$  which interpolates  $(A, B)$  and thus renders it a non-gap. Avoiding this particular example, an  $(\omega_1, \omega_1)$  pregap  $(A, B)$  in  $\mathcal{P}(\mathbb{N}) / \text{Fin}$  is called *destructible* if there is an  $\aleph_1$  preserving poset which forces that  $(A, B)$  is not a gap.

The statement “all  $(\omega_1, \omega_1)$  gaps in  $\mathcal{P}(\mathbb{N}) / \text{Fin}$  are indestructible” is in fact a Ramsey theoretic statement which is closely analogous to Souslin’s Hypothesis. This becomes clear when one considers the characterization of destructibility in Theorem 1.3 below. We credit Theorems 1.1 and 1.3 to Kunen (1976) and Woodin (1984), respectively. The elegant Ramsey theoretic presentation is due to Todorćević (1989). See e.g. Todorćević (1989), Scheepers (1993), and Todorćević and Farah (1995) for proofs.

When working with pregarps in  $\mathcal{P}(\mathbb{N}) / \text{Fin}$  one often works with representatives (i.e. subsets of  $\mathbb{N}$ ) of the equivalence classes. With every pair of families  $a_i, b_i \subseteq \mathbb{N}$  indexed by  $i \in I$  we associate a partition  $[I]^2 = K_0 \cup K_1$  via

$$(1) \quad \{i, j\} \in K_0 \quad \text{iff} \quad (a_i \cap b_j) \cup (a_j \cap b_i) = \emptyset.$$

In the case where (the equivalence classes of)  $(a_i, b_i : i \in I)$  is a pregap in  $\mathcal{P}(\mathbb{N}) / \text{Fin}$ , we may assume—in order to avoid trivialities and thereby obtain more concise results—that the representatives have been chosen so that

$$\star \quad a_i \cap b_i = \emptyset \quad \text{for all } i \in I.$$

The following theorem characterizes  $(\omega_1, \omega_1)$  gaps in  $\mathcal{P}(\mathbb{N}) / \text{Fin}$ .

**Theorem 1.1** (Kunen). *For every  $(\omega_1, \omega_1)$  pregap in  $\mathcal{P}(\mathbb{N}) / \text{Fin}$  with representatives chosen satisfying condition  $\star$  the following are equivalent:*

- (a)  $(a_\alpha, b_\alpha : \alpha < \omega_1)$  is a gap.
- (b) There is no uncountable 0-homogeneous subset of  $\omega_1$ .
- (c) The poset of all finite 1-homogeneous subsets of  $\omega_1$  has the ccc.

*Remark 1.2.* Only the case  $I = \omega_1$  was considered because in general the theorem is false, although for  $(\kappa, \kappa)$  pregaps with  $\kappa$  is regular and uncountable, it does generalize by replacing “uncountable” with “cardinality  $\kappa$ ” and “ccc” with “ $\kappa$ -cc”.

By switching the colors 0 and 1 one obtains a characterization of destructibility.

**Theorem 1.3** (Woodin). *For every  $(\omega_1, \omega_1)$  pregap in  $\mathcal{P}(\mathbb{N}) / \text{Fin}$  with representatives satisfying  $\star$  the following are equivalent:*

- (a)  $(a_\alpha, b_\alpha : \alpha < \omega_1)$  is destructible.
- (b) There is no uncountable 1-homogeneous subset of  $\omega_1$ .
- (c) The poset of all finite 0-homogeneous subsets of  $\omega_1$  has the ccc.

Considering condition (b) of Theorem 1.1, condition (c) of Theorem 1.3 says that there is a poset with the ccc forcing that  $(a_\alpha, b_\alpha : \alpha < \omega_1)$  is not a gap. A Souslin tree itself has the ccc and forces an  $\omega_1$ -branch through itself, which is analogous to forcing an interpolation of a gap. And condition (b) of Theorem 1.3 can be viewed in analogy with the property that Souslin trees have no uncountable antichains. With this in mind, we provisionally refer to destructible and *Souslin*  $(\omega_1, \omega_1)$  pregaps interchangeably, until further definitions are made in Section 1.5.

*Remark 1.4.* It is a theorem of Kunen (1976) (see also e.g. Scheepers (1993)) that for cardinals  $\kappa$  and  $\lambda$ , every  $(\kappa, \lambda)$  pregap in  $\mathcal{P}(\mathbb{N}) / \text{Fin}$  with either  $\kappa \neq \omega_1$  or  $\lambda \neq \omega_1$  can be interpolated by a ccc poset. E.g. when  $\kappa = \lambda \neq \omega_1$ , the corresponding condition Theorem 1.3(c) (i.e. replace “ $\omega_1$ ” with “ $\kappa$ ”) holds for any such pregap.

The analogy goes further when one considers the influence of four additional set theoretic axioms: the existence of a diamond sequence ( $\diamond$ ); the Continuum Hypothesis (CH); the principle  $(*)$  for ideals of countable subsets of  $\omega_1$ , a consequence of PFA which is consistent with CH and entails many combinatorial consequences of PFA (see Abraham and Todorćević (1997) and Todorćević (2000)); and Martin’s Axiom for  $\aleph_1$  many dense subsets ( $\text{MA}_{\aleph_1}$ ). Jensen proved that  $\diamond$  implies the existence of a Souslin tree, while Todorćević proved in (Dow, 1995) (he took credit for this in private communication) that  $\diamond$  implies the existence of a destructible  $(\omega_1, \omega_1)$  gap. It is a theorem of Jensen (Devlin and Johnsbråten, 1974) that SH is consistent with CH, while Abraham–Todorćević (1997) proved that  $(*)$  implies both SH and “all  $(\omega_1, \omega_1)$  gaps are indestructible” (and in particular “all  $(\omega_1, \omega_1)$  gaps are indestructible” is consistent with CH). It is a theorem of Solovay–Tennenbaum (1971) that  $\text{MA}_{\aleph_1}$  implies SH, while it is a theorem of Kunen (1976) that  $\text{MA}_{\aleph_1}$  implies that there are no Souslin  $(\omega_1, \omega_1)$  gaps in  $\mathcal{P}(\mathbb{N}) / \text{Fin}$ . These results are summarized in the **trivial** column of Tables 1 and 2.

The analogy can be carried still further by considering the influence of Cohen forcing. It is a theorem of Shelah (1984) that Cohen forcing (i.e. adding one Cohen real) always produces a Souslin tree, while it is a theorem of Todorćević (see Todorćević and Farah (1995)) that Cohen forcing produces a Souslin gap.

Table 1: *Souslin's Hypothesis*

Axiom	Measurable algebra		
	<b>trivial</b>	<b>separable</b>	<b>nonseparable</b>
$\diamond$	False	False	False
CH	Undecided	Undecided	Undecided
ZFC	Undecided	Undecided	Undecided
(*)	True	True	True
$\text{MA}_{\aleph_1}$	True	True	True

Table 2: *All gaps are indestructible*

Axiom	Measurable algebra		
	<b>trivial</b>	<b>separable</b>	<b>nonseparable</b>
$\diamond$	False	False	False
CH	Undecided	False	False
ZFC	Undecided	Undecided	False
(*)	True	Undecided	False
$\text{MA}_{\aleph_1}$	True	True	False

This naturally leads us at once to consider the influence of the other fundamental forcing notion, random forcing: a separable measurable algebra (adding one random real), or more generally forcing with an arbitrary measurable algebra (possibly nonseparable, adding uncountably many random reals).

Let us recall here that a Boolean algebra  $\mathcal{B}$  is *measurable* if there is a function  $\mu : \mathcal{B} \rightarrow [0, \infty)$  such that  $(\mathcal{B}, \mu)$  is a measure algebra. The measure  $\mu$  has a naturally associated metric  $d_\mu$  on  $\mathcal{B}$ , where the distance between  $a$  and  $b$  is  $d_\mu(a, b) = \mu(a \triangle b)$ . The measure algebra is *(non)separable* if the metric topology on  $\mathcal{B}$  is (non)separable. Note that given a measurable  $\mathcal{B}$  separability is independent of the choice of measure  $\mu$ ; this follows from Maharam's Theorem which is discussed below.

The influence on Souslin's Hypothesis of forcing with some measurable algebra over a model satisfying any of the above axioms, is known and summarized in Table 1. It is a theorem of Laver (1987) that under  $\text{MA}_{\aleph_1}$ , forcing with any measurable algebra preserves SH. And it is a theorem of the author (2007a) that assuming (\*), forcing with any measurable algebra preserves SH. Note that the other rows follow from these results because Souslin trees are preserved by forcing notions satisfying property  $K$  (i.e. Knaster's chain condition), and in particular by any measurable algebra.

It is a result of the author (2003) that under  $\text{MA}_{\aleph_1}$ , all gaps are indestructible in any forcing extension by a separable measurable algebra. In (2004b) the author proves that under CH, adding one random real produces a Souslin gap (the CH row and **separable** column of Table 2); and this is the first place where

the analogy breaks down, proving that: *Souslin's Hypothesis is consistent with the existence of a Souslin  $(\omega_1, \omega_1)$  gap in  $\mathcal{P}(\mathbb{N}) / \text{Fin}$ .* The following relatively old question of Woodin has received the attention of a number of authors.

**Question 1** (Woodin). *Does  $\text{MA}_{\aleph_1}$  imply that all  $(\omega_1, \omega_1)$  gaps in  $\mathcal{P}(\mathbb{N}) / \text{Fin}$  are indestructible in any forcing extension by a measurable algebra?*

The main result of this paper, the “False” in the ZFC row of the **nonseparable** column of Table 2, answers Question 1 negatively and it completes the picture by allowing us to fill in the remainder of Table 2:

**Theorem 1.** *Let  $\mathcal{R}$  be a nonseparable measurable algebra. Then some condition in  $\mathcal{R}^+$  forces that there exists a Souslin  $(\omega_1, \omega_1)$  gap in  $\mathcal{P}(\mathbb{N}) / \text{Fin}$ .*

### 1.3 Terminology

We have been calling  $(\mathcal{B}, \leq)$  a Boolean algebra, to indicate that  $(\mathcal{B}, \leq)$  is a partial ordering with minimum and maximum elements 0 and 1, resp., such that every two elements  $x, y \in \mathcal{B}$  have both an infimum  $x \wedge y$  and a supremum  $x \vee y$  and also has a complement  $-x = \bigvee \{z \in \mathcal{B} : x \wedge z = 0\}$ .<sup>1</sup> This agrees with the usual definition of a Boolean algebra as a structure of the form  $(\mathcal{B}, \wedge, \vee, -, 0, 1)$ , in that  $(\mathcal{B}, \leq)$ , with  $x \leq y$  defined by  $x - y = 0$ , has the above properties iff this structure satisfies the axioms of a Boolean algebra.

Recall that a *measure space* is a triple  $(X, \Sigma, \nu)$  where  $\Sigma$  is a  $\sigma$ -algebra of subsets of  $X$  consisting of the  $\nu$ -measurable sets,  $\nu : \Sigma \rightarrow [0, \infty]$  is a function with  $\nu(\emptyset) = 0$  and

$$(2) \quad \nu \left( \bigcup_{n=0}^{\infty} E_n \right) = \sum_{n=0}^{\infty} \nu(E_n)$$

whenever  $(E_n : n \in \mathbb{N})$  is a sequence of pairwise disjoint  $\nu$ -measurable sets. A *measure algebra* is a pair  $(\mathcal{B}, \mu)$  where  $\mathcal{B}$  is a  $\sigma$ -complete Boolean algebra and  $\mu : \mathcal{B} \rightarrow [0, \infty]$  is a function with  $\mu(0) = 0$  and  $\mu(\bigvee_{n=0}^{\infty} a_n) = \sum_{n=0}^{\infty} \mu(a_n)$  whenever  $a_m \wedge a_n = 0$  for all  $m \neq n$ . A *probability space* is a measure space  $(X, \Sigma, \nu)$  with  $\nu(X) = 1$ ; similarly, a *probability algebra* is a measure algebra  $(\mathcal{R}, \mu)$  with  $\mu(1) = 1$ . A  $\sigma$ -finite measure space is a measure space with  $\nu$ -measurable sets  $(E_n : n \in \mathbb{N})$  such that  $X = \bigcup_{n=0}^{\infty} E_n$  and  $\nu(E_n) < \infty$  for all  $n$ ; a  $\sigma$ -finite measure algebra is defined analogously.

For a measure space  $(X, \nu)$  we write  $\mathcal{N}_\nu$  for the ideal of all  $A \subseteq X$  with  $\nu(A) = 0$ . These sets are called *null* (or *negligible*). The *measure algebra of a measure space*  $(X, \Sigma, \nu)$  is the quotient  $\Sigma / \mathcal{N}_\nu$  with the well-defined measure  $\mu([E]) = \nu(E)$ .

We recall some further definitions we shall need. The *additivity* of a measure algebra  $\text{add}(\mu)$  is the smallest cardinal  $\kappa$  for which there is a pairwise incompatible family  $\mathcal{A} \subseteq \mathcal{R}$  of cardinality  $\kappa$  such that the equality

$$(3) \quad \mu \left( \bigvee \mathcal{A} \right) = \bigvee_{a \in \mathcal{A}} \mu(a)$$

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<sup>1</sup>We use the notation  $x - y$  to abbreviate  $x \wedge (-y)$ .

fails. Thus  $\text{add}(\mu) \geq \aleph_1$ . A similar definition can be made for a measure space. Note that the additivity of the ideal  $\mathcal{N}_\nu$  satisfies  $\text{add}(\mathcal{N}_\nu) = \text{add}(\nu)$ . For some cardinal  $\kappa$ , we say that a measure  $\nu$  is  $\kappa$ -*additive* to indicate that  $\text{add}(\nu) \geq \kappa$ . A measurable set  $E \subseteq X$  is called an *atom* if it is not null and every  $F \subseteq E$  with  $\mu(F) < \mu(E)$  is null; similarly, for a nonzero element of a measure algebra.

## 1.4 Applications

### 1.4.1 Souslin trees versus gaps

An immediate consequence is a simple construction<sup>2</sup> of a model satisfying SH which also has a destructible gap:

**Corollary 1.5.** *Adding uncountably many random reals to any model of ZFC satisfying  $\text{MA}_{\aleph_1}$  gives a model of “there are no Souslin trees” and “there is a Souslin  $(\omega_1, \omega_1)$  gap in  $(\mathcal{P}(\mathbb{N}) / \text{Fin}, \subseteq^*)$ ”.*

### 1.4.2 Consequences of RVM

Another corollary is that the classical hypothesis that the Lebesgue measure can be extended to all subsets of the real line  $\mathbb{R}$ , implies the existence of a destructible  $(\omega_1, \omega_1)$  gap in  $\mathcal{P}(\mathbb{N}) / \text{Fin}$ . This is an immediate consequence of Theorem 1 and known absoluteness results for forcing extensions by a large enough measure algebra, from a real-valued measurable cardinal. We thankfully acknowledge Stevo Todorćević for suggesting Corollary 1.9 (in May 2002).

This absoluteness is in fact an axiomatization of random forcing. We state a theorem (Theorems 1.7) to this effect, without proof, some variation of which is folklore. It says that if a statement of reasonable complexity is forced to hold in the extension by a large enough measurable algebra, then this statement is a consequence of the axiom that the Lebesgue measure extends to all subsets of the real line. This axiom is often called RVM.

We will not give the proof here, or even a full explanation of the terminology. For a complete proof, and also a variation of Theorem 1.7, with other aspects of the mathematical background explained, we refer the reader to the supplement to this paper (Hirschorn, 2007b); and for further reading we also suggest the paper (Caicedo, 2006). Recall that an *atomlessly measurable cardinal*  $\kappa$  is an uncountable cardinal carrying an atomless  $\kappa$ -additive probability measure with domain  $\mathcal{P}(\kappa)$ . It is a classical theorem of Ulam (1930) that RVM is equivalent to the existence of an atomlessly measurable cardinal  $\kappa$ , and that  $\kappa$  is larger than the least weakly inaccessible cardinal but  $\kappa \leq 2^{\aleph_0}$ .

**Notation 1.6.** For a set  $X$ , we write  $(\mathcal{R}_X, \mu_X)$  for the measure algebra of the measure space  $\{0, 1\}^X$  with its Haar probability measure.

A word on Maharam’s Theorem is also in order here. It states roughly that every  $\sigma$ -finite measure algebra (cf. §1.3), up to an isomorphism, has a simple decomposition into measure algebras of the form  $(\mathcal{R}_\theta, \mu_\theta)$ , where  $\theta$  is some cardinal. A measurable algebra (cf. §1.2) is *homogeneous* in the forcing

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<sup>2</sup>The point is that it is relatively difficult to construct a model of  $(*)$  and CH, and this consistency result was originally proved by adding one random real to such a model.

sense iff it is isomorphic to  $\mathcal{R}_{(\theta)}$  for some cardinal  $\theta$ . The *Maharam type* of such a measurable algebra is the cardinal  $\theta$ . By Maharam's Theorem, every  $\sigma$ -finite measurable algebra  $\mathcal{R}$  has a dense set of  $z \in \mathcal{R}^+$  such that  $\mathcal{R}_z = \{x \in \mathcal{R} : x \leq z\}$  is homogeneous. Note that a homogeneous measurable algebra is nonseparable iff its Maharam type is at least  $\aleph_1$ . For more information on Maharam's Theorem, see also e.g. Fremlin (2001) and Hirschorn (2004a).

**Theorem 1.7.** *Suppose  $\kappa$  is an atomlessly measurable cardinal. If  $\varphi(x, y)$  is a  $\Pi_1$  formula and  $a \in H_{\kappa^+}$  is a parameter such that every homogeneous measurable algebra of large enough Maharam type forces the statement*

$$(4) \quad \exists x \in H_{\kappa^+} \varphi(x, \check{a}),$$

*then  $V \models \ulcorner \exists x \varphi(x, a) \urcorner$ .*

*Remark 1.8.* There is an important theorem of Gitik and Shelah (1989) (see also Fremlin (1993)) giving a lower bound on the Maharam type of the measure algebras associated with atomlessly measurable cardinals: *Let  $X$  be a set. If  $\nu : \mathcal{P}(X) \rightarrow [0, 1]$  is an atomless probability measure then the measure algebra  $\mathcal{P}(X) / \mathcal{N}_\nu$  has Maharam type at least  $\min\{\text{add}(\nu)^{+\omega}, 2^{\text{add}(\nu)}\}$ .* (Moreover they proved in Gitik and Shelah (2001) that the Maharam type is  $2^{2^{\aleph_0}}$  in the case  $X = 2^{\aleph_0}$  and  $\text{add}(\nu) = 2^{\aleph_0}$ .) The “large enough Maharam type” in Theorem 1.7 is the Maharam type of  $\mathcal{P}(\kappa) / \mathcal{N}_\nu$ , and thus is at least  $\min\{\kappa^{+\omega}, 2^\kappa\}$ .

**Corollary 1.9** (ZFC+RVM). *There exists a Souslin  $(\omega_1, \omega_1)$  gap in  $\mathcal{P}(\mathbb{N})/\text{Fin}$ .*

*Proof.* Using Woodin's Theorem (Theorem 1.3), it is routine to obtain a  $\Pi_1$  formula  $\varphi(x, y, a)$  so that  $\varphi(x, y, \omega_1)$  holds iff  $(x, y)$  forms a Souslin  $(\omega_1, \omega_1)$  gap in  $\mathcal{P}(\mathbb{N}) / \text{Fin}$ . By the assumption that the Lebesgue measure on the real line can be extended to a measure whose domain is all of  $\mathcal{P}(\mathbb{R})$ , there exists an atomlessly measurable cardinal  $\kappa$ . Since  $(\omega_1, \omega_1)$  gaps are objects of  $H_{\aleph_2}$ , by Theorem 1, every homogeneous measure algebra of Maharam type at least  $\aleph_1$  forces  $\ulcorner \exists x, y \in H_{\kappa^+} \varphi(x, y, \check{\omega}_1) \urcorner$ . Therefore, from Theorem 1.7 we conclude that there exists  $(x, y)$  satisfying  $\varphi(x, y, \omega_1)$ , completing the proof.  $\square$

## 1.5 Souslin gaps

If  $(A, B)$  is a gap, not necessarily linear, where the cardinalities of  $A$  and  $B$  are both at most  $\aleph_1$ , then forcing with a poset which collapses  $\aleph_1$  will still interpolate the gap. Hence, it makes sense to extend the definition of *destructible* to include this situation.

Broadening our scope to include arbitrary pregaps in  $\mathcal{P}(\mathbb{N})/\text{Fin}$ , Theorem 1.3 is no longer true. In condition (c), the poset of finite 0-homogeneous subsets will force an interpolation of an uncountable subgap, but if the gap is nonlinear it will not necessarily interpolate the whole gap. We would like to isolate this condition (c), as the Souslin property.

**Definition 1.10.** A pregap  $(A, B)$  in  $\mathcal{P}(\mathbb{N}) / \text{Fin}$  is called *Souslin* if every uncountable family  $\mathcal{F}$  of finite subsets of  $A \times B$  (i.e. finite subpregaps of  $(A, B)$ ) has an  $\aleph_1$  preserving forcing extension where there exists  $d \in \mathcal{P}(\mathbb{N}) / \text{Fin}$  such that

$$(5) \quad \{F \in \mathcal{F} : d \text{ interpolates } F\} \text{ is uncountable.}$$

Note we need only consider families  $\mathcal{F}$  of size  $\aleph_1$ , and that by a  $\Delta$ -system argument we can assume  $\mathcal{F}$  is pairwise disjoint.

This Souslin property of gaps can be characterized in terms of representative subsets of  $\mathbb{N}$ . For this it will be convenient to reconsider the requirement  $\star$ . Suppose  $(a_i, b_i : i \in I)$  is an indexing of two sequences of members of  $\mathcal{P}(\mathbb{N})$ . Instead of partitioning the subsets of cardinality two  $[I]^2$  as in (1), we can define partitions of the pairs  $I^2 = L_0^k \cup L_1^k$  and generalize by adding a parameter  $k \in \mathbb{N}$ , as in

$$(6) \quad (i, j) \in L_0^k \quad \text{iff} \quad (a_i \cap b_j \setminus k) \cup (a_j \cap b_i \setminus k) = \emptyset,$$

and we write  $L_0$  and  $L_1$  for  $L_0^0$  and  $L_1^0$ , respectively. Then the condition  $\star$  is equivalent to “ $L_0(i, i)$  for all  $i \in I$ ”, and assuming this is satisfied, a subset of  $I$  is  $K_0$ -homogeneous iff it is  $L_0$ -homogeneous. Notice that the quantification in the next theorem is significantly different than in Theorems 1.1 and 1.3.

**Theorem 1.11.** *Let  $(A, B)$  be a pregap in  $\mathcal{P}(\mathbb{N}) / \text{Fin}$ . Then the following are equivalent:*

- (a)  $(A, B)$  is Souslin.
- (b) For every choice  $(a_i, b_i : i \in I)$  of representatives the associated poset of all finite  $L_0$ -homogeneous subsets of  $I$  has the ccc.
- (c) There exists an indexing  $(a_i, b_i : i \in I)$  of representatives such that for every  $k$  the associated poset of all finite  $L_0^k$ -homogeneous subsets of  $I$  has the ccc.

*Proof.* (a)  $\rightarrow$  (b): Suppose that  $(A, B)$  is Souslin and  $(a_i, b_i : i \in I)$  is a choice of representatives. Let  $J_\alpha \in \text{Fin}_I$  ( $\alpha < \omega_1$ ) be a family of 0-homogeneous finite subsets of  $I$ . By going to an uncountable subset we can assume that they are all the same size, say  $J_\alpha = \{i_0^\alpha, \dots, i_{n-1}^\alpha\}$  for all  $\alpha$ . Now go to an  $\aleph_1$  preserving forcing extension with  $d \subseteq \mathbb{N}$  such that

$$(7) \quad X = \{\alpha < \omega_1 : d \text{ interpolates } \{(a_i, b_i) : i \in J_\alpha\}\} \text{ is uncountable.}$$

By going to an uncountable subset of  $X$  we can assume there is a  $k \in \mathbb{N}$  such that  $a_i \setminus k \subseteq d$  and  $b_i \setminus k \subseteq d^c$  for all  $i \in J_\alpha$  for all  $\alpha \in X$ . Finally, we can pick  $\alpha \neq \beta$  in  $X$  such that both  $\{(a_i, b_i) : i \in J_\alpha\}$  and  $\{(a_i, b_i) : i \in J_\beta\}$  have the same trace on  $k$ . Then  $(J_\alpha \cup J_\beta)^2 \subseteq K_0$  because for all  $l, m = 0, \dots, n-1$ ,

$$(8) \quad \begin{aligned} a_{i_l^\alpha} \cap b_{i_m^\beta} &= (a_{i_l^\alpha} \cap b_{i_m^\beta} \cap k) \cup (a_{i_l^\alpha} \cap b_{i_m^\beta} \setminus k) \\ &= a_{i_l^\alpha} \cap b_{i_m^\beta} \cap k \cup \emptyset \\ &= a_{i_l^\alpha} \cap b_{i_m^\alpha} \cap k \\ &= \emptyset, \end{aligned}$$

as  $a_{i_l^\alpha} \cap b_{i_m^\alpha} = \emptyset$  for all  $l, m$  by homogeneity.

(b)  $\rightarrow$  (c): Take *any* indexing  $(a_i, b_i : i \in I)$  of representatives. Suppose  $\mathcal{F}$  is an uncountable family of finite  $L_0^k$ -homogeneous subsets of  $I$  for some fixed  $k \in \mathbb{N}$ . Applying condition (b) to the indexing  $(a_i \setminus k, b_i \setminus k : i \in I)$  of



representatives, since each  $F \in \mathcal{F}$  is 0-homogeneous for the associated partition, it follows that there exist  $F \neq G$  in  $\mathcal{F}$  such that  $(F \cup G)^2 \subseteq L_0^k$ .

(c)  $\rightarrow$  (a): Suppose that  $(a_i, b_i : i \in I)$  is an indexing of representatives satisfying the hypothesis of (c). For each  $F \in \mathcal{F}$  choose a finite  $J_F \subseteq I$  such that  $F \subseteq \{a_i : i \in J_F\} \times \{b_i : i \in J_F\}$ , and then find  $k_F \in \mathbb{N}$  large enough so that

$$(9) \quad J_F^2 \subseteq L_0^{k_F}.$$

Pick  $k \in \mathbb{N}$  large enough so that  $\mathcal{F}_0 = \{F \in \mathcal{F} : k_F = k\}$  is uncountable. It follows from condition (c) that the poset of all finite  $\mathcal{H} \subseteq \mathcal{F}_0$  such that  $\bigcup_{F \in \mathcal{H}} J_F$  is  $L_0^k$ -homogeneous has the ccc, and therefore it can have only countably many atoms and thus forces an uncountable subset  $\mathcal{G} \subseteq \mathcal{F}_0$  each member of which is interpolated by  $\bigcup_{F \in \mathcal{G}} \bigcup_{i \in J_F} a_i$ .  $\square$

Pregaps have been called *Luzin* in the literature (e.g. Farah (2004)) when there exists an uncountable  $K_1$ -homogeneous subset of  $I$  for some indexing of representatives satisfying  $\star$ . By Theorem 1.11 (a)  $\rightarrow$  (b), every Souslin pregap in  $\mathcal{P}(\mathbb{N}) / \text{Fin}$  is non-Luzin, and thus for pregaps of size at most  $\aleph_1$

$$(10) \quad \text{destructible} \rightarrow \text{Souslin} \rightarrow \text{non-Luzin}.$$

In the realm of linear gaps, Souslin gaps are precisely the non-Luzin ones, and for linear gaps of size  $\aleph_1$  all three notions coincide by Theorem 1.3. It may be worth revisiting this concept in a separate article.

## 1.6 Gaps in $(L^0(\nu) / \text{Fin}, \subseteq_{\text{ae}}^*)$

Let  $(\mathcal{R}, \mu)$  be the measure algebra of some  $\sigma$ -finite measure space  $(X, \Sigma, \nu)$ . In considering  $\mathcal{R}$ -names for gaps in  $\mathcal{P}(\mathbb{N}) / \text{Fin}$ , we are led to consider gaps in another Boolean algebra. Recall that a *random variable on  $X$  with codomain  $S$* , where  $S$  is some topological space, is an almost everywhere defined function  $f$ , measurable for the completion of  $\mu$ , taking values in  $S$ ; in other words,  $f$  is defined on a conegligible subset of  $X$  and there is a conegligible  $C \subseteq X$  such that for every open  $U \subseteq S$ ,  $f^{-1}[U] \cap C \in \Sigma$ . The family of all equivalence classes of random variables on  $X$  with codomain  $\mathbb{R}$  over the equivalence relation  $f(x) = g(x)$  almost everywhere, is the standard space  $L^0(\nu)$  from functional analysis. We are interested here in taking  $\mathcal{P}(\mathbb{N})$  to be our reals; hence, we define  $L^0(\nu) / \text{Fin}$  to be the family of all equivalence classes of random variables on  $X$  with codomain  $\mathcal{P}(\mathbb{N})$  modulo the relation  $f(x) \triangle g(x)$  is finite almost everywhere, and order  $L^0(\nu) / \text{Fin}$  by defining  $[f] \subseteq_{\text{ae}}^* [g]$  if  $f(x) \subseteq^* g(x)$  almost everywhere. This clearly defines a Boolean algebra. By a *random gap* we mean a (pre)gap in a Boolean algebra of this form.

We have seen that Maharam's Theorem says that for purposes of forcing, we can restrict our attention to measure algebras of the form  $(\mathcal{R}_{(\theta)}, \mu_{(\theta)})$  for some cardinal  $\theta$ . And it is well known (e.g. Scott (1967)) that there is a direct correspondence between  $\mathcal{R}_{(\theta)}$ -names  $\dot{a}$  for a subset of  $\mathbb{N}$  and random variables  $f$  on the underlying measure space  $\{0, 1\}^\theta$  with codomain  $\mathcal{P}(\mathbb{N})$ , via

$$(11) \quad \|f(\dot{r}) = \dot{a}\| = 1,$$

where  $\dot{r}$  is a name for the generic object in  $\{0, 1\}^\theta$ . Of course  $f$  must be interpreted correctly in the forcing extension to make sense of (11), but the point is that every measurable function  $f : \{0, 1\}^\theta \rightarrow \mathcal{P}(\mathbb{N})$  is equal almost everywhere to a Baire function<sup>3</sup>  $g : \{0, 1\}^\theta \rightarrow \mathcal{P}(\mathbb{N})$ , and every such Baire function can be coded by a countable sequence of ordinals so that every suitable model containing this sequence has a correct interpretation of  $g$  as a Baire function from  $\{0, 1\}^\theta$  into  $\mathcal{P}(\mathbb{N})$ . Henceforth, we will dot random variables (e.g.  $\dot{a}$ ) to emphasize this correspondence.

For two of these random variables  $\dot{a}$  and  $\dot{b}$ , clearly  $\|\dot{a}(\dot{r}) \subseteq^* \dot{b}(\dot{r})\| = 1$  iff  $[\dot{a}] \subseteq_{\text{ae}}^* [\dot{b}]$  in  $L^0(\mu_{(\theta)}) / \text{Fin}$ . Thus a pregap in  $(L^0(\mu_{(\theta)}) / \text{Fin}, \subseteq_{\text{ae}}^*)$  is the same thing as an  $\mathcal{R}_{(\theta)}$ -name for a pregap in  $(\mathcal{P}(\mathbb{N}) / \text{Fin}, \subseteq^*)$ . And a pregap in  $L^0(\mu_{(\theta)}) / \text{Fin}$  is a gap iff when viewed as an  $\mathcal{R}_{(\theta)}$ -name for a pregap it is a gap in  $\mathcal{P}(\mathbb{N}) / \text{Fin}$  with positive probability. We also would like to extend the correspondence to the notion of Souslin gaps, keeping in mind Theorem 1.11.

**Definition 1.12.** A pregap  $(A, B)$  in  $L^0(\nu) / \text{Fin}$  is *Souslin* if there is an enumeration  $(\dot{a}_\alpha, \dot{b}_\alpha : \alpha < \omega_1)$  of representatives, so that for every sequence  $(E_\xi, \Gamma_\xi : \xi < \omega_1)$  and every  $k \in \mathbb{N}$ , where  $E_\xi \in \Sigma$  has positive finite measure,  $\Gamma_\xi \subseteq \omega_1$  is finite, and

$$(12) \quad \Pr \left[ \bigcup_{\alpha, \beta \in \Gamma_\xi} \dot{a}_\alpha \cap \dot{b}_\beta \setminus k = \emptyset \mid E_\xi \right] = 1 \quad \text{for all } \xi$$

(equivalently,  $\bigcup_{\alpha, \beta \in \Gamma_\xi} \dot{a}_\alpha(x) \cap \dot{b}_\beta(x) \setminus k = \emptyset$  almost everywhere in  $E_\xi$ ), there exist  $\xi \neq \eta$  such that  $\nu(E_\xi \cap E_\eta) \neq 0$  and

$$(13) \quad \Pr \left[ \bigcup_{\alpha \in \Gamma_\xi} \bigcup_{\beta \in \Gamma_\eta} (\dot{a}_\alpha \cap \dot{b}_\beta \setminus k) \cup (\dot{a}_\beta \cap \dot{b}_\alpha \setminus k) = \emptyset \mid E_\xi \cap E_\eta \right] \neq 0.$$

Note that in the case where  $\nu = \mu_{(\theta)}$ , when identifying random variables  $\dot{a}$  on  $\{0, 1\}^\theta$  with codomain  $\mathcal{P}(\mathbb{N})$  with the  $\mathcal{R}_{(\theta)}$ -name  $\dot{a}(\dot{r})$  for a subset of  $\mathbb{N}$ , then replacing each  $E_\xi$  with  $x_\xi \in \mathcal{R}^+$ , condition (12) becomes

$$(14) \quad x_\xi \wedge \bigvee_{\alpha, \beta \in \Gamma_\xi} \|\dot{a}_\alpha \cap \dot{b}_\beta \setminus k \neq \emptyset\| = 0 \quad \text{for all } \xi,$$

and condition (13) becomes, with  $x_\xi \wedge x_\eta \neq 0$ ,

$$(15) \quad x_\xi \wedge x_\eta - \bigvee_{\alpha \in \Gamma_\xi} \bigvee_{\beta \in \Gamma_\eta} \|(\dot{a}_\alpha \cap \dot{b}_\beta \setminus k) \cup (\dot{a}_\beta \cap \dot{b}_\alpha \setminus k) \neq \emptyset\| \neq 0.$$

Thus, by Theorem 1.11, a pregap in  $L^0(\mu_{(\theta)})$  is Souslin iff it is forced to be a Souslin pregap with probability one when viewed as an  $\mathcal{R}_{(\theta)}$ -name.

For a measure space  $(X, \Sigma, \nu)$ ,  $\nu$  is called nonseparable if the measure algebra of the measure space is nonseparable. Thus we obtain the following equivalent formulation of Theorem 1.

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<sup>3</sup>We mean Baire measurable function, where the Baire sets are members of the smallest  $\sigma$ -algebra for which every continuous function is measurable. This is the  $\sigma$ -algebra generated by the clopen sets in the case of a Cantor cube  $\{0, 1\}^X$ .

**Theorem 2.** *There exists an  $(\omega_1, \omega_1)$  Souslin gap in  $(L^0(\nu) / \text{Fin}, \subseteq_{\text{ae}}^*)$  whenever  $\nu$  is a nonseparable  $\sigma$ -finite measure.*

Let us note that gaps in  $L^0(\nu) / \text{Fin}$  are a generalization of gaps in  $\mathcal{P}(\mathbb{N}) / \text{Fin}$ , because if  $(X, \Sigma, \nu)$  is a trivial measure space with one element of measure one, then  $(L^0(\nu) / \text{Fin}, \subseteq_{\text{ae}}^*)$  and  $(\mathcal{P}(\mathbb{N}) / \text{Fin}, \subseteq^*)$  are isomorphic. Finally, we notice that the existence of an  $(\omega_1, \omega_1)$  gap in  $L^0(\nu) / \text{Fin}$  for any  $\sigma$ -finite measure  $\nu$ , follows from Hausdorff's Theorem on the existence of these gaps in  $\mathcal{P}(\mathbb{N}) / \text{Fin}$  applied in the appropriate random forcing extension.

### 1.6.1 Destructible gaps in $L^0(\nu) / \text{Fin}$

The fact that there is no gap  $(A, B)$  in  $\mathcal{P}(\mathbb{N}) / \text{Fin}$  with both  $A$  and  $B$  countable generalizes to gaps in  $L^0(\nu) / \text{Fin}$  for any  $\sigma$ -finite measure  $\nu$ . Therefore, an  $(\omega_1, \omega_1)$  gap in  $L^0(\nu) / \text{Fin}$  is interpolated by collapsing  $\aleph_1$ . Thus we can define a destructible  $(\omega_1, \omega_1)$  gap in  $L^0(\nu) / \text{Fin}$  as one which can be interpolated by a forcing which preserves  $\aleph_1$ . Note that, in  $L^0(\nu) / \text{Fin}$ , destructible  $(\omega_1, \omega_1)$  gaps and Souslin  $(\omega_1, \omega_1)$  gaps are two different things! Indeed in proving Theorem 2 we shall construct a Souslin gap which is indestructible. On the other hand, in Hirschorn (2004b), an  $(\omega_1, \omega_1)$  gap in  $L^0(\nu) / \text{Fin}$  with  $\nu$  separable, is constructed with the aid of the Continuum Hypothesis, and this gap is both destructible and Souslin; moreover, this gap can be destroyed by a poset with the property  $K$ , which cannot happen to a gap in  $\mathcal{P}(\mathbb{N}) / \text{Fin}$ . It is not hard to prove every destructible  $(\omega_1, \omega_1)$  gap in  $L^0(\nu) / \text{Fin}$  must be Souslin.

## 2 Measure theoretic characterizations

There are some simple measure theoretic characterizations, or more precisely necessary and/or sufficient conditions, of when an  $\mathcal{R}$ -name for a pregap names a Souslin gap and related phenomenon. This is our explanation for the complex behavior observed in the interactions between gaps in  $\mathcal{P}(\mathbb{N}) / \text{Fin}$  and random forcing. For example, in Hirschorn (2004b) it is shown that for a separable probability algebra  $(\mathcal{R}, \nu)$ , an  $\mathcal{R}$ -name  $(\dot{a}_\alpha, \dot{b}_\alpha : \alpha < \omega_1)$  for an pregap in  $\mathcal{P}(\mathbb{N}) / \text{Fin}$  is forced to be Souslin with positive probability, if

$$(16) \quad \mu(\|\dot{a}_\alpha \cap \dot{b}_\beta \neq \emptyset\|) < \frac{1}{2} \quad \text{for all } \alpha, \beta < \omega_1,$$

(or less than some other constant below 1). The statement (16) is far simpler than the definition of destructibility or its Ramsey theoretic characterization in Theorem 1.3. Note that unlike Theorem 1.3, the characterization in equation (16) is by no means equivalent to destructibility; in the case of the trivial measure algebra it implies that the pregap is a non-gap.

We have also obtained in Hirschorn (2002) a simple necessary condition for an  $\mathcal{R}$ -name for a pregap to satisfy the *Hausdorff property*, the strengthening of indestructibility to  $\{\alpha < \beta : a_\alpha \cap b_\beta \subseteq k\}$  is finite for all  $k \in \mathbb{N}$ , for all  $\beta < \omega_1$ , which was satisfied by Hausdorff's original construction of an  $(\omega_1, \omega_1)$  gap. It states that if the equation (32) holds for some  $h \in c_0$  (i.e.  $\lim_{n \rightarrow \infty} h(n) = 0$ ), then we do not have a name for a gap satisfying Hausdorff's condition. In fact, in Hirschorn (2002) an  $\mathcal{R}$ -name (with  $\mathcal{R}$  nonseparable) for an indestructible

gap not satisfying Hausdorff's condition is constructed by satisfying (32) for an  $h \in \ell^1$  (i.e.  $\sum_{n=0}^{\infty} h(n) < \infty$ ). Therefore, the characterization of the Souslin property in Theorem 3 is not valid with the hypothesis (32) alone.

We were unable to find a sufficient condition for the Souslin property with an arbitrary measurable algebra, purely by putting constraints on the measures of various events. This lead us to probabilistic considerations in our goal to obtain a characterization of Souslin gaps in  $L^0(\nu) / \text{Fin}$  (this is achieved in Theorem 3 below).

Recall that a family  $\{\mathcal{S}_i : i \in I\}$  of subsets of some measure algebra  $(\mathcal{R}, \mu)$  is *stochastically independent* if  $\mu(x \wedge y) = \mu(x)\mu(y)$  for all  $x \in \mathcal{S}_i$  and  $y \in \mathcal{S}_j$ , for all  $i \neq j$  in  $I$ . And two elements  $x, y \in \mathcal{R}$  are called stochastically independent when  $\{\{x\}, \{y\}\}$  is a stochastically independent family.

The following basic result of probability theory is not used for the characterization of a Souslin gap, but is needed later on in the actual construction of an  $(\omega_1, \omega_1)$  Souslin gap in  $L^0(\nu) / \text{Fin}$  (a proof can be found in (Hirschorn, 2007b)).

**Lemma 2.1.** *Let  $\{\{x_i, y_i\} : i \in I\}$  be a stochastically independent family such that  $x_i \wedge y_i = 0$  for all  $i \in I$ . Then for all  $A, B \subseteq I$ ,  $\mu(\bigvee_{i \in A} x_i \wedge \bigvee_{i \in B} y_i) \leq \mu(\bigvee_{i \in A} x_i) \mu(\bigvee_{i \in B} y_i)$  with equality iff either  $A \cap B = \emptyset$  or  $x_i = 1$  or  $y_i = 1$  for some  $i \in A \cup B$ . In other words,  $\bigvee_{i \in A} x_i$  is unfavourable for  $\bigvee_{i \in B} y_i$ .*

Recall that for  $A \subseteq X$ , a subset  $S$  of the Cantor cube  $\{0, 1\}^X$  is called *determined by coordinates in A* or *A-determined* if for all  $y, z \in \{0, 1\}^X$ ,

$$(17) \quad y \upharpoonright A = z \upharpoonright A \quad \text{implies} \quad y \in S \quad \text{iff} \quad z \in S.$$

And  $x \in \mathcal{R}_{(X)}$  is *determined by coordinates in A* or *A-determined* if  $x$  is a member of the natural identification of  $\mathcal{R}_{(A)}$  with a subalgebra of  $\mathcal{R}_{(X)}$  (cf. Notation 1.6). Thus  $x \in \mathcal{R}_{(X)}$  is *A-determined* iff it has an *A-determined* representative  $S \subseteq \{0, 1\}^X$ . The set of *coordinates determining x* is the minimum  $A \subseteq X$  which determines  $x$ . Such an  $A$  always exists; indeed it is the set of all  $\xi \in X$  such that

$$(18) \quad \pi_{\langle \xi, 0 \rangle}(x \wedge \langle \xi, 0 \rangle) \neq \pi_{\langle \xi, 1 \rangle}(x \wedge \langle \xi, 1 \rangle),$$

where  $\pi_{\langle \xi, i \rangle} : (\mathcal{R}_{(X)})_{\langle \xi, i \rangle} \rightarrow \mathcal{R}_{(X \setminus \{\xi\})}$  is the natural isomorphism as in Corollary 2.3, and  $\langle \xi, i \rangle$  denotes the element of the measure algebra represented by  $\{z \in \{0, 1\}^X : z(\xi) = i\}$ . Recall that every member of  $\mathcal{R}_{(X)}$  is represented by an  $F_\sigma$  Baire set, and as such is determined by countably many coordinates. The set of *coordinates determining S* where  $S \subseteq \mathcal{R}_{(X)}$  is the union of the coordinates that determine each member of  $S$ .

We say that a collection  $\{\mathcal{S}_i : i \in I\}$  of subsets of  $\mathcal{R}_{(X)}$  is *independently determined* if there is a family  $\{A_i : i \in I\}$  of pairwise disjoint subsets of  $X$  such that  $x$  is  $A_i$ -determined for all  $x \in \mathcal{S}_i$ , i.e.  $\mathcal{S}_i \subseteq \mathcal{R}_{(A_i)}$ , for all  $i \in I$ . We also say that two elements  $x, y \in \mathcal{R}_{(X)}$  are *independently determined* if the family  $\{\{x\}, \{y\}\}$  is independently determined, and similarly we can say that  $x$  and  $S \subseteq \mathcal{R}_{(X)}$  are *independently determined* to indicate that  $\{\{x\}, S\}$  is independently determined.

Note that if  $\mathcal{F}$  is an independently determined family of subsets then  $\mathcal{F}$  is stochastically independent; however, conversely, two stochastically independent members of the measure algebra may fail to be disjointly determined.

For a finite partial function  $s : X \dashrightarrow \{0, 1\}$ , we let  $[s] = [s]_{(X)} \in \mathcal{R}_{(X)}$  be the equivalence class of the basic clopen set

$$(19) \quad \{z \in \{0, 1\}^X : z \supseteq s\}.$$

In some contexts,  $[s]$  will denote the clopen set (19) instead. Let  $\text{Fin}(X, \{0, 1\})$  denote the collection of all finite partial functions from  $X$  into  $\{0, 1\}$ .

**Lemma 2.2.** *Let  $(\mathcal{R}, \mu) = (\mathcal{R}_{(\theta)}, \mu_{(\theta)})$  for some infinite cardinal  $\theta$ , and let  $t \in \text{Fin}(\theta, \{0, 1\})$ . Then every  $x \in \mathcal{R}_{[t]}$  has a unique  $\theta \setminus \text{dom}(t)$ -determined  $y \in \mathcal{R}$  such that  $x = y \wedge [t]$ .*

**Corollary 2.3.** *There is a unique measure algebra isomorphism  $\pi : \mathcal{R}_{[t]} \rightarrow \mathcal{R}_{(\theta \setminus \text{dom}(t))}$  such that  $\pi([s]) = [s \setminus t]$  for all  $s \in \text{Fin}(\theta, \{0, 1\})$  compatible with  $t$ .*

*Proof.* Let  $\pi(x)$  be the identification of the element of  $\mathcal{R}$  given by Lemma 2.2 with an element of  $\mathcal{R}_{(\theta \setminus \text{dom}(t))}$ .  $\square$

Maharam's Theorem gives an isomorphism between  $\mathcal{R}_{[t]}$  and  $\mathcal{R}_{(\theta \setminus \text{dom}(t))}$ , but the point of the preceding corollary was to identify the natural one.

## 2.1 A chain condition for conditional probabilities

There is a classical chain condition for probability algebras due to Gillis (1936), which entails that every uncountable subset  $\mathcal{A}$  of a probability algebra, in which every element has measure greater than some  $\delta$ , contains an uncountable subset  $\mathcal{B} \subseteq \mathcal{A}$  where  $\mu(x \wedge y) > \delta^2$  for every two elements  $x$  and  $y$  in  $\mathcal{B}$ . What we would like here is an analogue for conditional probabilities:

$$(20) \quad \mu(x \mid y) = \frac{\mu(x \wedge y)}{\mu(y)},$$

i.e. for uncountable sequences  $x_\alpha, y_\alpha \in \mathcal{R}^+$  ( $\alpha < \omega_1$ ) such that  $\mu(x_\alpha \mid y_\alpha) > \delta$  for all  $\alpha$ , there exists an uncountable  $X \subseteq \omega_1$  such that  $\mu(x_\alpha \wedge x_\beta \mid y_\alpha \wedge y_\beta) > \delta^2$  for all  $\alpha \in X$ . Although this is false as stated (cf. (Hirschorn, 2007b)), if the  $y_\beta$ 's are of a simple enough form then it becomes valid (Theorem 2.4).

**Theorem 2.4.** *Let  $(\mathcal{R}, \mu) = (\mathcal{R}_{(\theta)}, \mu_{(\theta)})$  for some cardinal  $\theta$ . If  $\mathcal{A}$  is an uncountable family of pairs  $(x, s)$  where  $x \in \mathcal{R}$ ,  $s \in \text{Fin}(\theta, \{0, 1\})$ , and  $\mu(x \mid [s]) > \delta$  for all  $(x, s) \in \mathcal{A}$ , then there is an uncountable  $\mathcal{B} \subseteq \mathcal{A}$  such that  $\mu(x \wedge y \mid [s] \wedge [t]) > \delta^2$  for all  $(x, s), (y, t) \in \mathcal{B}$ .*

*Proof.* Let  $(x_\alpha, s_\alpha)$  ( $\alpha < \omega_1$ ) enumerate a subset of  $\mathcal{A}$ . Find an uncountable  $X \subseteq \omega_1$  such that  $\{\text{dom}(s_\alpha) : \alpha \in X\}$  forms a  $\Delta$ -system with root  $\Gamma \subseteq \theta$ , and

$$(21) \quad \begin{aligned} |s_\alpha \setminus \Gamma| &= m \\ s_\alpha \upharpoonright \Gamma &= t \end{aligned} \quad \text{for all } \alpha \in X.$$

Denote  $\mu_{(\theta \setminus \Gamma)}$  by  $\rho$ , and write  $\pi = \pi_{[t]}$  for the isomorphism from Corollary 2.3. Since the scalar correction of  $\mu$  to a probability measure on  $\mathcal{R}_{[t]}$  is given by  $\mu(\cdot) / \mu([t])$ , we have  $\rho(\pi(x)) = \frac{\mu(x)}{\mu([t])}$  for all  $x \in \mathcal{R}_{[t]}$ . However, by cancellation the scalar multiple of a measure has the same conditional probabilities

as the original measure. Hence, it suffices to find an uncountable  $X' \subseteq X$  such that

$$(22) \quad \rho(\pi(x_\alpha) \wedge \pi(x_\beta) \mid \pi([s_\alpha]) \wedge \pi([s_\beta])) > \delta^2 \quad \text{for all } \alpha, \beta \in X'.$$

Furthermore, the hypothesis translates to  $\rho(\pi(x_\alpha \wedge [s_\alpha])) > \delta \rho(\pi([s_\alpha]))$  for all  $\alpha$ . All of the  $\rho(\pi([s_\alpha]))$ 's for  $\alpha \in X$  are equal to some fixed  $\sigma > 0$ —namely,  $\sigma = 2^{-m}$  by (21). Thus by Gillis' Theorem, there is an uncountable  $X' \subseteq X$  such that

$$(23) \quad \rho(\pi(x_\alpha \wedge [s_\alpha]) \wedge \pi(x_\beta \wedge [s_\beta])) > \delta^2 \sigma^2 \quad \text{for all } \alpha, \beta \in X'.$$

But by the  $\Delta$ -system construct,  $\pi([s_\alpha]) = [s_\alpha \setminus t]$  is determined independently of  $\pi([s_\beta]) = [s_\beta \setminus t]$  for all  $\alpha \neq \beta$  in  $X$ , and thus by stochastic independence,  $\rho(\pi([s_\alpha]) \wedge \pi([s_\beta])) = \sigma^2$ . This establishes (22).  $\square$

## 2.2 Continuous representatives

In Hirschorn (2000) we observed that every member of  $L^0(\mu_{(\theta)}) / \text{Fin}$  has a continuous representative (actually the entire paper deals only with  $\theta = \omega$ , but the proof of this fact applies to arbitrary  $\theta$ ). Here we make additional specifications on the representatives. The existence of continuous representatives is essentially the fact that the collection of equivalence classes of clopen sets is dense in the metric topology on  $\mathcal{R}_{(\theta)}$  (in other words, for every measurable set  $A \subseteq \{0, 1\}^\theta$  and every  $\varepsilon > 0$  there exists a clopen  $C \subseteq \{0, 1\}^\theta$  such that  $\mu(A \triangle C) < \varepsilon$ ).

**Lemma 2.5.** *For every pair of random variables  $\dot{a}, \dot{b}$  on  $\{0, 1\}^\theta$  with codomain  $\mathcal{P}(\mathbb{N})$ , and every  $h : \mathbb{N} \rightarrow [0, 1]$  with*

$$(24) \quad \mu(\|n \in \dot{a} \cap \dot{b}\|) + h(n) > 0 \quad \text{for all } n,$$

*there exist continuous functions  $\dot{c}, \dot{d} : \{0, 1\}^\theta \rightarrow \mathcal{P}(\mathbb{N})$  such that*

$$(a) \quad \|\dot{c} \cap \dot{d}\| = \emptyset,$$

*and for all  $n$ ,*

$$(b) \quad \|n \in \dot{c}\| \text{ and } \|n \in \dot{d}\| \text{ are both determined by the set of coordinates which determines } \{\|n \in \dot{a}\|, \|n \in \dot{b}\|\},$$

$$(c) \quad \mu(\|n \in \dot{a} \triangle \dot{c}\|) < \mu(\|n \in \dot{a} \cap \dot{b}\|) + h(n),$$

$$(d) \quad \mu(\|n \in \dot{b} \triangle \dot{d}\|) < \mu(\|n \in \dot{a} \cap \dot{b}\|) + h(n).$$

*Proof.* For each  $n$ : Let  $A_n \subseteq \theta$  be the set of coordinates determining  $\{\|n \in \dot{a}\|, \|n \in \dot{b}\|\}$ . In case  $A_n$  is finite,  $\|n \in \dot{a}\|$  and  $\|n \in \dot{b}\|$  both have clopen representatives  $C_n, D_n \subseteq \{0, 1\}^\theta$ , respectively. Otherwise, when  $A_n$  is infinite, working in the space  $\{0, 1\}^{A_n}$ , we can find a measurable  $E_n \subseteq \|n \in \dot{a} \cap \dot{b}\|$  which is  $A_n$ -determined with measure

$$(25) \quad \mu(E_n) = \frac{\mu(\|n \in \dot{a} \cap \dot{b}\|)}{2}.$$

Requirement (24) allows us to find  $A_n$ -determined clopen sets  $C_n$  and  $D'_n$  such that

$$(26) \quad \mu\left(C_n \triangle \left(\|n \in \dot{a}\| \setminus (\|n \in \dot{b}\| \setminus E_n)\right)\right) < \frac{\mu(\|n \in \dot{a} \cap \dot{b}\|) + h(n)}{6},$$

$$(27) \quad \mu\left(D'_n \triangle (\|n \in \dot{b}\| \setminus E_n)\right) < \frac{\mu(\|n \in \dot{a} \cap \dot{b}\|) + h(n)}{6},$$

i.e. (26) and (27) hold for any representatives of  $\|n \in \dot{a}\|$  and  $\|n \in \dot{b}\|$ . Then since  $\|n \in \dot{a}\| \cap (\|n \in \dot{b}\| \setminus E_n) = \|n \in \dot{a} \cap \dot{b}\| \setminus E_n$ , which has measure  $\mu(\|n \in \dot{a} \cap \dot{b}\|) / 2$  by (25), recalling the metric  $d_\mu$  (cf. page 4), the triangle inequality gives

$$(28) \quad \mu(C_n \triangle \|n \in \dot{a}\|) < \frac{2\mu(\|n \in \dot{a} \cap \dot{b}\|) + h(n)}{3}.$$

And from (26) and (27) we have  $\mu(C_n \cap D'_n) < [\mu(\|n \in \dot{a} \cap \dot{b}\|) + h(n)] / 3$ , and thus letting  $D_n = D'_n \setminus C_n$  yields

$$(29) \quad \begin{aligned} \mu(D_n \triangle \|n \in \dot{b}\|) &\leq \mu(D_n \triangle (\|n \in \dot{b}\| \setminus E_n)) \\ &\quad + \mu((\|n \in \dot{b}\| \setminus E_n) \triangle \|n \in \dot{b}\|) \\ &\leq \mu(D_n \triangle D'_n) + \mu(D'_n \triangle (\|n \in \dot{b}\| \setminus E_n)) + \mu(E_n) \\ &< \left(\frac{1}{3} + \frac{1}{6} + \frac{1}{2}\right) [\mu(\|n \in \dot{a} \cap \dot{b}\|) + h(n)] \\ &= \mu(\|n \in \dot{a} \cap \dot{b}\|) + h(n), \end{aligned}$$

using the triangle inequality with  $d_\mu$ .

Now the functions  $\dot{c}, \dot{d} : \{0, 1\}^\theta \rightarrow \mathcal{P}(\mathbb{N})$  given by  $\dot{c}(x) = \{n : x \in C_n\}$  and  $\dot{d}(x) = \{n : x \in D_n\}$  are continuous and as needed.  $\square$

*Remark 2.6.* Note that if in Lemma 2.5,  $\sum_{n=0}^\infty \mu(\|n \in \dot{a} \cap \dot{b}\|) < \infty$ , then  $\|[\dot{a}] = [\dot{c}]\| = 1$ , i.e.  $[\dot{a}] = [\dot{c}]$  in  $L^0(\mu_{(\theta)}) / \text{Fin}$ , because

$$(30) \quad \|[\dot{a}] \neq [\dot{c}]\| = \bigwedge_{k=0}^\infty \bigvee_{n=k}^\infty \|n \in \dot{a} \triangle \dot{c}\|,$$

and obviously  $\|[\dot{b}] = [\dot{d}]\| = 1$  too.

### 2.3 Characterization

The following Theorem 3 identifies some Souslin random gaps.

**Theorem 3.** *Let  $\mu = \mu_{(\theta)}$  for some cardinal  $\theta$ . Suppose  $\{\dot{a}_\alpha : \alpha < \omega_1\}$  and  $\{\dot{b}_\alpha : \alpha < \omega_1\}$  are two families of representatives of members of  $L^0(\mu) / \text{Fin}$ . If*

$$(31) \quad \left\{ \bigcup_{\alpha < \omega_1} \{\|n \in \dot{a}_\alpha\|, \|n \in \dot{b}_\alpha\|\} : n \in \mathbb{N} \right\} \text{ is independently determined,}$$

and for some  $h \in \ell^1$ ,

$$(32) \quad \mu(\|n \in \dot{a}_\alpha \cap \dot{b}_\beta\|) \leq h(n) \quad \text{for all } \alpha, \beta < \omega_1 \text{ and all } n \in \mathbb{N},$$

then  $(\{\dot{a}_\alpha : \alpha < \omega_1\}, \{\dot{b}_\alpha : \alpha < \omega_1\})$  forms a Souslin pregap.

Note that condition (32) implies that  $\{\dot{a}_\alpha : \alpha < \omega_1\}$  is orthogonal to  $\{\dot{b}_\alpha : \alpha < \omega_1\}$  because  $[\dot{a}_\alpha] \wedge [\dot{b}_\beta] = 0$  in  $L^0(\mu) / \text{Fin}$  is equivalent to  $\|\dot{a}_\alpha \cap \dot{b}_\beta\|$  is infinite  $= 0$ , and

$$(33) \quad \|\dot{a}_\alpha \cap \dot{b}_\beta \text{ is infinite}\| = \bigwedge_{k=0}^{\infty} \bigvee_{n=k}^{\infty} \|n \in \dot{a}_\alpha \cap \dot{b}_\beta\|.$$

Condition (31) is a very demanding requirement. If we consider an  $(\omega_1, \omega_1)$  gap in  $L^0(\mu) / \text{Fin}$ , i.e. a linear gap, then we can relax this to  $(\{\|n \in \dot{a}_\alpha\|, \|n \in \dot{b}_\alpha\|\} : n \in \mathbb{N})$  is independently determined sequence for every  $\alpha$  separately. We feel that such requirements are not very natural and that with the right probability theory (concerning countable collections of random variables) a natural characterization, say  $\{\{\|n \in \dot{a}_\alpha\|, \|n \in \dot{b}_\alpha\|\} : n \in \mathbb{N}\}$  is a stochastically independent family, or even some further weakening, is obtainable.

### 2.3.1 Proof of Theorem 3

The proof begins by choosing a sequence  $(A^n : n \in \mathbb{N})$  of pairwise disjoint subsets of  $\theta$  such that

$$(34) \quad \|n \in \dot{a}_\alpha\| \text{ and } \|n \in \dot{b}_\alpha\| \text{ are both } A^n\text{-determined for all } n, \text{ for all } \alpha.$$

Without loss of generality assume that  $h > 0$ . Then for each  $\alpha$ , we apply Lemma 2.5 to the pair  $(\dot{a}_\alpha, \dot{b}_\alpha)$ , with  $h_\alpha(n) = h(n) - \mu(\|n \in \dot{a}_\alpha \cap \dot{b}_\alpha\|)$ , to obtain a pair  $(\dot{c}_\alpha, \dot{d}_\alpha)$  of continuous functions as in the conclusion of the Lemma. Note that we have

$$(35) \quad \begin{aligned} \mu(\|n \in \dot{c}_\alpha \cap \dot{d}_\beta\|) &\leq \mu(\|n \in \dot{a}_\alpha \cap \dot{b}_\beta\|) + \mu(\|n \in \dot{c}_\alpha \setminus \dot{a}_\alpha\|) \\ &\quad + \mu(\|n \in \dot{d}_\beta \setminus \dot{b}_\beta\|) \\ &< h(n) + (\mu(\|n \in \dot{a}_\alpha \cap \dot{b}_\alpha\|) + h_\alpha(n)) \\ &\quad + (\mu(\|n \in \dot{a}_\beta \cap \dot{b}_\beta\|) + h_\beta(n)) \\ &= 3h(n) \end{aligned}$$

for all  $n$ , for all  $\alpha, \beta < \omega_1$ . For each  $\alpha$ , for each  $n \in \mathbb{N}$ , letting

$$(36) \quad y_\alpha^n = \|n \in \dot{c}_\alpha\| \quad \text{and} \quad z_\alpha^n = \|n \in \dot{d}_\alpha\|,$$

by continuity we can write  $y_\alpha^n = \bigvee_{s \in C_\alpha^n} [s]$  and  $z_\alpha^n = \bigvee_{s \in D_\alpha^n} [s]$  where  $C_\alpha^n, D_\alpha^n \subseteq \text{Fin}(A^n, \{0, 1\})$  are both finite, using clause (b) of Lemma 2.5. Thus

$$(37) \quad B_\alpha^n = \bigcup_{s \in C_\alpha^n \cup D_\alpha^n} \text{dom}(s) \subseteq A^n$$

is finite.



By Remark 2.6, it suffices to prove that  $(\dot{c}_\alpha, \dot{d}_\alpha : \alpha < \omega_1)$  is Souslin. Suppose that  $k \in \mathbb{N}$ ,  $(x_\xi : \xi < \omega_1)$  is a sequence in  $\mathcal{R}^+$  and  $(\Gamma_\xi : \xi < \omega_1)$  is a sequence of finite subsets of  $\omega_1$  such that

$$(38) \quad x_\xi \wedge \bigvee_{\alpha, \beta \in \Gamma_\xi} \|\dot{c}_\alpha \cap \dot{d}_\beta \setminus k \neq \emptyset\| = 0 \quad \text{for all } \xi.$$

By going to an uncountable subsequence, we can assume that  $|\Gamma_\xi| = m$  for all  $\xi$ .

Let  $\varepsilon > 0$  be given. For each  $\xi < \omega_1$ , choose  $s_\xi \in \text{Fin}(\theta, \{0, 1\})$  such that

$$(39) \quad \mu([s_\xi] \wedge x_\xi) > \sqrt{1 - \frac{\varepsilon}{2}} \cdot \mu([s_\xi]).$$

Since the  $A_n$ 's are pairwise disjoint, if we choose a large enough  $p_\xi \in \mathbb{N}$ , then since  $h \in \ell^1$  and by equation (37),

$$(40) \quad \sum_{n=p_\xi}^{\infty} h(n) < \frac{\varepsilon}{24m^2},$$

$$(41) \quad \text{dom}(s_\xi) \cap \bigcup_{\alpha \in \Gamma_\xi} \bigcup_{n=p_\xi}^{\infty} B_\alpha^n = \emptyset.$$

It clearly follows from (39) that there is a  $t_\xi \supseteq s_\xi$  where

$$(42) \quad \text{dom}(t_\xi) = \bigcup_{\alpha \in \Gamma_\xi} \bigcup_{n < p_\xi} B_\alpha^n,$$

$$(43) \quad \mu(x_\xi \mid [t_\xi]) > \sqrt{1 - \frac{\varepsilon}{2}}.$$

By going to an uncountable subset  $X \subseteq \omega_1$ , we arrange that  $p_\xi = p$  for all  $\xi \in X$ , and that  $\mu([t_\xi]) = \tau$  for all  $\xi \in X$ . Choose  $l \geq p$  large enough so that

$$(44) \quad \sum_{n=l}^{\infty} h(n) < \frac{\tau^2 \varepsilon}{24m^2}.$$

And for each  $\xi \in X$ , put

$$(45) \quad \Omega_\xi = \bigcup_{\alpha \in \Gamma_\xi} \bigcup_{n < l} B_\alpha^n.$$

By going to an uncountable subsequence, we obtain  $X' \subseteq X$  such that  $\{\Omega_\xi : \xi \in X'\}$  forms a  $\Delta$ -system, say with root  $\Omega$ . Let  $(\gamma^i : i < o)$  be the strictly increasing enumeration of  $\Omega$ . By further refinement, we can furthermore assume that

$$(46) \quad (\{(C_\alpha^n, D_\alpha^n : n < l) : \alpha \in \Gamma_\xi\}, t_\xi, (\gamma^i : i < o)) \\ \cong (\{(C_\beta^n, D_\beta^n : n < l) : \beta \in \Gamma_\eta\}, t_\eta, (\gamma^i : i < o))$$

for all  $\xi, \eta \in X'$ , meaning that there is a finite partial injection  $g$  on  $\omega_1$  such that if every instance of each ordinal  $\zeta$  appearing in the structure  $(\{(C_\alpha^n, D_\alpha^n :$

$n < l) : \alpha \in \Gamma_\xi\}, t_\xi, (\gamma^i : i < o))$  is replaced with  $g(\zeta)$  then  $(\{(C_\beta^n, D_\beta^n : n < l) : \beta \in \Gamma_\eta\}, t_\eta, (\gamma^i : i < o))$  is obtained, i.e.  $\{(C_\beta, D_\beta : n < l) : \beta \in \Gamma_\eta\} = \{(\{s \circ g^{-1} : s \in C_\alpha^n\}, \{s \circ g^{-1} : s \in D_\alpha^n\} : n < l) : \alpha \in \Gamma_\xi\}$ ,  $t_\eta = t_\xi \circ g^{-1}$  and  $g(\gamma^i) = \gamma^i$  for all  $i < o$ .

**Claim 2.7.**  $[t_\xi] \wedge [t_\eta] \wedge ((y_\alpha^n \wedge z_\beta^n) \vee (y_\beta^n \wedge z_\alpha^n)) = 0$  for all  $k \leq n < p$ , for all  $\alpha \in \Gamma_\xi$  and all  $\beta \in \Gamma_\eta$ , for all  $\xi, \eta \in X'$ .

*Proof.* Fix  $\xi, \eta \in X'$  and  $k \leq n < p$ . For every  $\alpha \in \Gamma_\xi$ , since  $\text{dom}(t_\xi) \supseteq B_\alpha^n$ ,  $[t_\xi] \wedge y_\alpha^n$  is either 0 or  $[t_\xi]$ , and  $[t_\xi] \wedge z_\alpha^n$  is either 0 or  $[t_\xi]$ . Similarly, for every  $\beta \in \Gamma_\eta$ ,  $[t_\eta] \wedge y_\beta^n$  and  $[t_\eta] \wedge z_\beta^n$  are both either 0 or  $[t_\eta]$ . Now fix  $\alpha \in \Gamma_\xi$  and  $\beta \in \Gamma_\eta$ . The isomorphism witnessing (46) for  $\xi$  and  $\eta$  maps  $(C_\delta^n, D_\delta^n)$  to  $(C_\beta^n, D_\beta^n)$  for some  $\delta \in \Gamma_\xi$ , so that both

$$(47) \quad [t_\eta] \wedge y_\beta^n = 0 \quad \text{iff} \quad [t_\xi] \wedge y_\delta^n = 0,$$

$$(48) \quad [t_\eta] \wedge z_\beta^n = 0 \quad \text{iff} \quad [t_\xi] \wedge z_\delta^n = 0.$$

However, by (38),  $x_\xi \wedge y_\alpha^n \wedge z_\delta^n = 0$  and  $x_\xi \wedge y_\delta^n \wedge z_\alpha^n = 0$ . And since  $[t_\xi] \wedge x_\xi \neq 0$ , this implies that at most one of  $[t_\xi] \wedge y_\alpha^n$  and  $[t_\xi] \wedge z_\delta^n$  is nonzero, and at most one of  $[t_\xi] \wedge y_\delta^n$  and  $[t_\xi] \wedge z_\alpha^n$  is nonzero. It follows that both  $[t_\xi] \wedge y_\alpha^n \wedge [t_\eta] \wedge z_\beta^n = 0$  and  $[t_\xi] \wedge z_\alpha^n \wedge [t_\eta] \wedge y_\beta^n = 0$ , proving the claim.  $\square$

Note that  $t_\xi$  is compatible with  $t_\eta$  for all  $\xi, \eta \in X'$  by (46), since  $t_\zeta \subseteq \Omega_\zeta$  for all  $\zeta \in X$  by (42) and (45).

**Claim 2.8.**  $[t_\xi \cup t_\eta]$  and  $\{\bigvee_{n=p}^{l-1} y_\alpha^n \wedge z_\beta^n, \bigvee_{n=p}^{l-1} y_\beta^n \wedge z_\alpha^n\}$  are independently determined for all  $\alpha \in \Gamma_\xi$  and  $\beta \in \Gamma_\eta$ , for all  $\xi \neq \eta$  in  $X'$ .

*Proof.* Clearly  $[t_\xi \cup t_\eta]$  is  $\text{dom}(t_\xi) \cup \text{dom}(t_\eta)$ -determined, and  $\{\bigvee_{n=p}^{l-1} y_\alpha^n \wedge z_\beta^n, \bigvee_{n=p}^{l-1} y_\beta^n \wedge z_\alpha^n\}$  is determined by coordinates in

$$(49) \quad \bigcup_{n=p}^{l-1} B_\alpha^n \cup \bigcup_{n=p}^{l-1} B_\beta^n.$$

But  $\text{dom}(t_\xi) \cap \bigcup_{n=p}^{l-1} B_\beta^n \subseteq \text{dom}(t_\xi) \cap \Omega_\eta$  is a subset of the root  $\Omega$ , while the isomorphism from (46) fixes everything in the root, and thus  $\text{dom}(t_\xi) \cap \bigcup_{n=p}^{l-1} B_\beta^n = \text{dom}(t_\xi) \cap \bigcup_{n=p}^{l-1} B_\alpha^n = \emptyset$  by (42). And similarly,  $\text{dom}(t_\eta) \cap \bigcup_{n=p}^{l-1} B_\alpha^n = \text{dom}(t_\eta) \cap \bigcup_{n=p}^{l-1} B_\beta^n = \emptyset$ . This establishes that  $\text{dom}(t_\xi) \cup \text{dom}(t_\eta)$  is disjoint from the set in (49), as required.  $\square$

Now using (43) in Theorem 2.4, we obtain  $\xi \neq \eta$  in  $X'$  such that

$$(50) \quad \mu([t_\xi \cup t_\eta] \wedge x_\xi \wedge x_\eta) > \left(1 - \frac{\varepsilon}{2}\right) \wedge \mu([t_\xi \cup t_\eta]).$$

And for all  $\alpha \in \Gamma_\xi$  and  $\beta \in \Gamma_\eta$ ,

$$\begin{aligned}
(51) \quad & \mu \left( [t_\xi \cup t_\eta] \wedge \bigvee_{n=k}^{\infty} y_\alpha^n \wedge z_\beta^n \right) \leq \mu \left( [t_\xi \cup t_\eta] \wedge \bigvee_{n=k}^{p-1} y_\alpha^n \wedge z_\beta^n \right) \\
& + \mu \left( [t_\xi \cup t_\eta] \wedge \bigvee_{n=p}^{l-1} y_\alpha^n \wedge z_\beta^n \right) + \mu \left( [t_\xi \cup t_\eta] \wedge \bigvee_{n=l}^{\infty} y_\alpha^n \wedge z_\beta^n \right) \\
& \leq 0 + \mu([t_\xi \cup t_\eta]) \wedge \mu \left( \bigvee_{n=p}^{l-1} y_\alpha^n \wedge z_\beta^n \right) \\
& + \mu \left( \bigvee_{n=l}^{\infty} y_\alpha^n \wedge z_\beta^n \right) \\
& \leq \mu([t_\xi \cup t_\eta]) \cdot \sum_{n=p}^{\infty} 3h(n) + \sum_{n=l}^{\infty} 3h(n) \\
& < \frac{\varepsilon \cdot \mu([t_\xi \cup t_\eta])}{4m^2},
\end{aligned}$$

where Claims 2.7 and 2.8 are used for the second inequality, (35) for the third, and (40) and (44) are used for the fourth inequality; and similarly

$$(52) \quad \mu \left( [t_\xi \cup t_\eta] \wedge \bigvee_{n=k}^{\infty} y_\beta^n \wedge z_\alpha^n \right) < \frac{\varepsilon \cdot \mu([t_\xi \cup t_\eta])}{4m^2}.$$

Thus the measure of  $[t_\xi \cup t_\eta] \wedge \bigvee_{\alpha \in \Gamma_\xi} \bigvee_{\beta \in \Gamma_\eta} \bigvee_{n=k}^{\infty} (y_\alpha^n \wedge z_\beta^n) \vee (y_\beta^n \wedge z_\alpha^n)$  is less than  $\frac{\varepsilon}{2} \cdot \mu([t_\xi \cup t_\eta])$ , which with (50) tells us that

$$\begin{aligned}
(53) \quad & \mu \left( [t_\xi \cup t_\eta] \wedge \left( x_\xi \wedge x_\eta \right. \right. \\
& \left. \left. - \bigvee_{\alpha \in \Gamma_\xi} \bigvee_{\beta \in \Gamma_\eta} \|(\dot{c}_\alpha \cap \dot{d}_\beta \setminus k) \cup (\dot{c}_\beta \cap \dot{d}_\alpha \setminus k) \neq \emptyset\| \right) \right) \\
& > (1 - \varepsilon) \cdot \mu([t_\xi \cup t_\eta]),
\end{aligned}$$

and in particular the condition (15) is satisfied. This proves that  $(\dot{c}_\alpha, \dot{d}_\alpha : \alpha < \omega_1)$  is Souslin, thereby completing the proof of Theorem 3.

### 3 $(\omega_1, \omega_1)$ Souslin gap

We conclude the paper with a proof of Theorem 2, and thus Theorem 1, by constructing an  $(\omega_1, \omega_1)$  Souslin gap in  $L^0(\mu)/\text{Fin}$  for  $\mu$  a nonseparable measure. As we have seen, we can assume that  $\mu = \mu_{(\theta)}$  for some uncountable cardinal  $\theta$ . Write  $\mathcal{R} = \mathcal{R}_{(\theta)}$ .

Define a mapping  $\phi : \theta \times \mathbb{N} \times \mathbb{Z} \rightarrow \text{Fin}(\theta, \{0, 1\})$  where  $\text{dom}(\phi(\alpha, i, j)) = [\alpha, \alpha + i)$  and the concatenation

$$(54) \quad \phi(\alpha, i, j)(\alpha + i - 1)\phi(\alpha, i, j)(\alpha + i - 2) \cdots \phi(\alpha, i, j)(\alpha)$$

is the base 2 representation of  $j \bmod 2^i$ . This can be expressed equivalently as

$$(55) \quad \phi(\alpha, i, j)(\alpha + k) = \frac{j \bmod 2^i}{2^k} \wedge 1 \quad \text{for all } k < i.$$

Define  $s_\alpha, t_\alpha \in \text{Fin}(\theta, \{0, 1\})^\mathbb{N}$  by

$$(56) \quad s_\alpha(n) = \phi(\omega \cdot \alpha + n \lceil \log_2(n+1) \rceil, \lceil \log_2(n+2) \rceil, 0),$$

$$(57) \quad t_\alpha(n) = \phi(\omega \cdot \alpha + n \lceil \log_2(n+1) \rceil, \lceil \log_2(n+2) \rceil, 1).$$

Define random variables  $\dot{c}_\alpha, \dot{d}_\alpha$  ( $\alpha < \theta$ ) by

$$(58) \quad \|n \in \dot{c}_\alpha\| = [s_\alpha],$$

$$(59) \quad \|n \in \dot{d}_\alpha\| = [t_\alpha].$$

First of all, note that since  $\log_2(n+2) \geq 1$  for all  $n$ , the third parameter in  $\phi$  ensures that

$$(60) \quad \|\dot{c}_\alpha \cap \dot{d}_\alpha = \emptyset\| = 1 \quad \text{for all } \alpha.$$

And for all  $\alpha \neq \beta$ , by stochastic independence,  $\mu(\|n \in \dot{c}_\alpha \cap \dot{d}_\beta\|) = 2^{-2 \lceil \log_2(n+2) \rceil} \leq \frac{1}{(n+2)^2}$  for all  $n$ , and this sequence is summable which implies that  $[\dot{c}_\alpha] \wedge [\dot{d}_\beta] = 0$  in  $L^0(\mu) / \text{Fin}$  (i.e.  $\dot{c}_\alpha \cap \dot{d}_\beta$  is finite with probability one).

On the other hand, the length  $\lceil \log_2(n+2) \rceil$  of the basic elements is short enough so that  $\mu(\|n \in \dot{c}_\alpha\|) = \mu(\|n \in \dot{d}_\alpha\|) = 2^{-\lceil \log_2(n+2) \rceil}$  and hence

$$(61) \quad \frac{1}{2n+4} < \mu(\|n \in \dot{c}_\alpha\|) \leq \frac{1}{n+2},$$

which is used to prove the following.

**Claim 3.1.** *With probability one: no subset of  $\mathbb{N}$  interpolates the two families  $\{\dot{c}_\alpha : \alpha < \theta\}$  and  $\{\dot{d}_\alpha : \alpha < \theta\}$ . In particular, they form a gap in  $L^0(\mu) / \text{Fin}$ .*

*Proof.* Suppose that  $r \in \{0, 1\}^\theta$  is an  $\mathcal{R}$ -generic object (over  $V$ ). Suppose that  $e \in V[r]$  is a subset of  $\mathbb{N}$  such that  $\dot{c}_\alpha[r] \subseteq^* e$  for all  $\alpha$ . Choose a countable  $J \subseteq \theta$  large enough so that  $e \in V[r \restriction I]$ , where  $I = \bigcup_{\alpha \in J} [\omega \cdot \alpha, \omega \cdot \alpha + \omega)$ . It follows from Kunen's Theorem (1984)—stating that for any  $K \subseteq \theta$  in  $V$ ,  $r \restriction K$  is an  $\mathcal{R}_{(K)}$ -generic object over  $V$  iff  $r \restriction K$  is  $\mathcal{R}_{(K)}$ -generic over  $V$  and  $r \restriction \theta \setminus K$  is  $\mathcal{R}_{(\theta \setminus K)}$ -generic over  $V[r \restriction K]$ —that  $V[r]$  is a forcing extension of  $V[r \restriction I]$  by the measure algebra  $\mathcal{S} = \mathcal{R}_{(\theta \setminus I)}$  taken in  $V[r \restriction I]$ , and that for every  $\alpha \notin J$ , the  $\mathcal{S}$ -names for  $\dot{c}_\alpha[r]$  and  $\dot{d}_\alpha[r]$ , have the same definitions as given in (58) and (59), respectively.

In  $V[r \restriction I]$ : Fix  $\alpha \notin J$ . Let  $\mathcal{J}$  be the set of all  $a \subseteq \mathbb{N}$  such that

$$(62) \quad \|a \cap \dot{c}_\alpha \text{ is finite}\| \neq 0.$$

Since

$$(63) \quad (n+1) \lceil \log_2(n+2) \rceil - n \lceil \log_2(n+1) \rceil \geq \lceil \log_2(n+2) \rceil$$

for all  $n \in \mathbb{N}$ ,  $\|n \in \dot{c}_\alpha\|$  ( $n \in \mathbb{N}$ ) is a stochastically independent sequence, and thus by Cauchy's criterion for infinite products,

$$\begin{aligned}
(64) \quad a \in \mathcal{J} & \text{ iff } \bigvee_{k=0}^{\infty} \bigwedge_{n \in a \setminus k} -\|n \in \dot{c}_\alpha\| \neq 0 \\
& \text{ iff } \sup_{k \in \mathbb{N}} \prod_{n \in a \setminus k} 1 - \mu(\|n \in \dot{c}_\alpha\|) = 1 \\
& \text{ iff } \sum_{n \in a} \mu(\|n \in \dot{c}_\alpha\|) < \infty \\
& \text{ iff } \sum_{n \in a} (n+1)^{-1} < \infty \\
& \text{ iff } a \in \mathcal{I}_{\frac{1}{n}}
\end{aligned}$$

where  $\mathcal{I}_{\frac{1}{n}}$  is the well known analytic ideal of all subsets of  $\mathbb{N}$  on which the function  $1/(n+1)$  is summable. In particular,  $\mathcal{J}$  is a nonprincipal ideal, which means that  $\|\dot{c}_\alpha\|$  is infinite  $\| = 1$ .

Since  $\|e \supseteq^* \dot{c}_\alpha\| \neq 0$ ,  $e \notin \mathcal{J}$ . However,  $\mathcal{J}$  is also the ideal of all subsets  $a \subseteq \mathbb{N}$  such that  $\|a \cap \dot{d}_\alpha\|$  is finite  $\| \neq 0$ . Hence the proof is complete, because we have shown that  $e \cap \dot{d}_\alpha[r]$  is infinite.  $\square$

Notice that by (63), for every  $n \in \mathbb{N}$ ,  $\bigcup_{\alpha < \omega_1} \{\|n \in \dot{c}_\alpha\|, \|n \in \dot{d}_\alpha\|\}$  is determined by the coordinates  $\bigcup_{\alpha < \omega_1} [\omega \cdot \alpha + n \lceil \log_2(n+1) \rceil, \omega \cdot \alpha + (n+1) \lceil \log_2(n+2) \rceil]$ , and thusly the families are independently determined for  $m \neq n$ , as condition (31) of Theorem 3 requires. Hence Theorem 3 entails that  $(\dot{c}_\alpha, \dot{d}_\alpha : \alpha < \omega_1)$  is Souslin by letting  $h \in \ell^1$  be given by  $h(n) = \frac{1}{(n+2)^2}$ , because

$$(65) \quad \mu(\|n \in \dot{c}_\alpha\|), \mu(\|n \in \dot{d}_\alpha\|) = 2^{-\lceil \log_2(n+2) \rceil} \leq \frac{1}{n+2} \quad \text{for all } n,$$

and therefore for all  $\alpha \neq \beta$ , by stochastic independence,

$$(66) \quad \mu(\|n \in \dot{c}_\alpha \cap \dot{d}_\beta\|) = \mu(\|n \in \dot{c}_\alpha\|) \mu(\|n \in \dot{d}_\beta\|) \leq h(n),$$

which with (60) assures condition (32).

Now we build an  $(\omega_1, \omega_1)$  Souslin gap  $(\dot{a}_\alpha, \dot{b}_\alpha : \alpha < \omega_1)$  where the  $[\dot{c}_\alpha] \subseteq_{\text{ae}}^* [\dot{a}_\alpha]$  and  $[\dot{d}_\alpha] \subseteq_{\text{ae}}^* [\dot{b}_\alpha]$  in  $L^0(\mu) / \text{Fin}$  for all  $\alpha < \omega_1$ . Therefore, since  $(\dot{c}_\alpha, \dot{d}_\alpha : \alpha < \omega_1)$  forms a gap by Claim 3.1,  $(\dot{a}_\alpha, \dot{b}_\alpha : \alpha < \omega_1)$  will automatically be a gap. The idea is to limit the augmentation of the measures of  $\|n \in \dot{c}_\alpha\|$  and  $\|n \in \dot{d}_\alpha\|$  so that Theorem 3 still applies.

The  $(\omega_1, \omega_1)$  gap we have been striving towards is obtained from an ‘ascending tower’ of sorts (espc. conditions (i) and (ii) below). Namely, a sequence  $T_\alpha \in \text{Fin}(\omega_1)^\mathbb{N}$  ( $\alpha < \omega_1$ ), where  $\text{Fin}(\omega_1)$  denotes the collection of all finite subsets of  $\omega_1$ , satisfying:

- (i)  $\alpha \in T_\alpha(n)$  for all  $n$ ,
- (ii)  $T_\xi(n) \subseteq T_\alpha(n)$  for all but finitely many  $n$ , for all  $\xi < \alpha$ ,
- (iii)  $|T_\alpha(n)| \leq \sqrt[3]{n+1}$  for all  $n$ .

Such a tower can easily be constructed by recursion on  $\alpha$ , by adding the requirement

$$(iv) \quad \lim_{n \rightarrow \infty} \frac{|T_\alpha(n)|}{\sqrt[3]{n+1}} = 0$$

to carry the recursion through.

Now  $(\dot{a}_\alpha, \dot{b}_\alpha : \alpha < \omega_1)$  is defined by

$$(67) \quad \|n \in \dot{a}_\alpha\| = \bigvee_{\xi \in T_\alpha(n)} [s_\xi(n)] \quad \text{and} \quad \|n \in \dot{b}_\alpha\| = \bigvee_{\xi \in T_\alpha(n)} [t_\xi(n)]$$

for all  $n \in \mathbb{N}$ , for all  $\alpha < \omega_1$ . Then in fact,  $\|\dot{a}_\alpha \supseteq \dot{c}_\alpha\| = 1$  and  $\|\dot{b}_\alpha \supseteq \dot{d}_\alpha\| = 1$  by (i), and with (ii) it follows that both  $\{\dot{a}_\alpha : \alpha < \omega_1\}$  and  $\{\dot{b}_\alpha : \alpha < \omega_1\}$  have order type  $\omega_1$ . Applying Lemma 2.1 to the independently determined family  $\{\{[s_\xi(n)], [t_\xi(n)]\} : \xi < \omega_1\}$ , with both of the subsets  $A$  and  $B$  equal to  $T_\alpha(n)$ , and using (65) with condition (iii) yields

$$(68) \quad \begin{aligned} \mu(\|n \in \dot{a}_\alpha \cap \dot{b}_\beta\|) &\leq \mu(\|n \in \dot{a}_\alpha\|) \mu(\|n \in \dot{b}_\beta\|) \\ &\leq \left( \frac{\sqrt[3]{n+1}}{n+2} \right)^2 \\ &< (n+1)^{-\frac{4}{3}} \end{aligned}$$

for all  $n \in \mathbb{N}$ , for all  $\alpha, \beta < \omega_1$ . Since this extension of the original nonlinear gap still satisfies (31), Theorem 3 applies with the function in  $\ell^1$  given by<sup>4</sup>

$$(69) \quad n \mapsto (n+1)^{-\frac{4}{3}}.$$

Therefore  $(\dot{a}_\alpha, \dot{b}_\alpha : \alpha < \omega_1)$  is an  $(\omega_1, \omega_1)$  Souslin gap in the Boolean algebra  $(L^0(\mu) / \text{Fin}, \subseteq_{\text{ae}}^*)$ , concluding the paper.

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<sup>4</sup>In this construction, any function which goes to infinity and is everywhere  $\geq 1$  can be used in place of  $\sqrt[3]{n+1}$  as long as the resulting function corresponding to (69) is in  $\ell^1$ .

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